J. Fluid Mech. (1995), *vol.* 294, *pp.* 367–390

Transition modes of rotating shallow water waves in a channel

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(Received 25 May 1994 and in revised form 10 February 1995)

Normal modes of shallow water waves in a channel wherein the Coriolis parameter and the depth vary in the spanwise direction are investigated based on the conservation of the number of zeros in an eigenfunction. As a result, it is generally shown that the condition for transition modes (Kelvin modes and mixed Rossby–gravity modes) to exist, besides Rossby and Poincaré modes, is determined only by boundary conditions. A Kelvin mode is interpreted as a modification of a Kelvin wave or a boundary wave along a closed boundary, and a mixed Rossby–gravity mode as a modification of an inertial oscillation or a boundary wave along an open boundary. Transition modes appearing in edge and continental-shelf waves, equatorial waves and free oscillations over a sphere are systematically understood by applying the theory in this paper.

1. Introduction

When we investigate wave phenomena in the atmosphere and in the oceans, we often treat problems of shallow water systems. It is well known that in a rotating shallow water system there exist families of Poincaré modes (inertial gravity modes) propagating in the positive and negative directions and a family of Rossby modes. Each family includes infinite modes, although the Rossby modes degenerate to steady geostrophic modes in special cases where the potential vorticity of the basic state f/H is constant.

Edge and continental-shelf waves considered in physical oceanography is one example of rotating shallow water systems. Among various theories on edge and continental-shelf waves, the simplest model is studied by Reid (1958), who dealt with a semi-infinite sloping shelf with constant gradient as shown in figure 1(*a*). In his model, besides the families of Poincaré and topographic Rossby modes, a Kelvin-wave-like mode exists, which behaves like a Rossby wave when the wavenumber is small while it resembles a gravity wave as the wavenumber increases. (In his paper, this mode was labelled as $j = 2$, $n = 0$.) In the model investigated by Iga (1993) wherein a slope with constant gradient is followed by a flat bottom with a rigid lid (figure 1*b*), a Kelvin-wave-like mode also exists (which he called the M_0-mode), though his intention was not to study waves trapped on a coast, but to interpret instability modes present in a two-layer model. As well as these cases, a Kelvin-wave-like mode usually exists in problems of edge and continental-shelf waves. Huthnance (1975) discussed generally the problems of waves over a continental shelf, and showed that under certain circumstances a single mode like a Kelvin wave exists besides families of Poincaré modes propagating in the positive and negative directions and a family of Rossby modes, although the two examples mentioned above are beyond his general theory.

However, in a similar problem of edge and continental-shelf waves examined by Mysak (1968), wherein the sloping shelf has a finite width that drops off vertically into

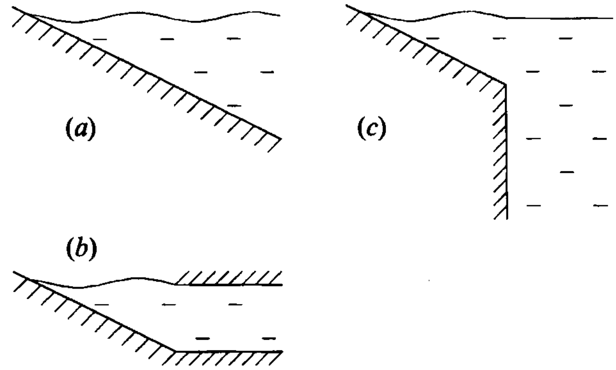

FIGURE 1. Various edge and continental-shelf wave problems. (a) A sloping shelf with constant gradient continues semi-infinitely. (b) A sloping shelf is followed by a flat bottom with a rigid lid. (c) A sloping shelf with a finite width drops off vertically into deep ocean.

infinitely deep ocean (figure 1 c), all modes are distinctly separated into either low- or high-frequency modes, and there is no mode which shifts from a low-frequency region to a high-frequency region as the wavenumber varies. These results show that a mode like a Kelvin wave exists in certain models, but does not in very similar other models. Why does a Kelvin-wave-like mode appear in certain cases and not in others that are only slightly different?

Incidentally, the mode of mixed Rossby–gravity waves present in the equatorial β-plane shifts from the region of Rossby waves to that of gravity waves with the change of wavenumber (Matsuno 1966). However, this mixed Rossby–gravity mode behaves in the opposite manner to the modes like a Kelvin wave mentioned above: it has features like a Rossby wave when the wavenumber is large and approaches inertial gravity waves as the wavenumber decreases.† Moreover, a mode called Kelvin wave also exists in equatorial waves.

Furthermore, in free oscillations on a rotating sphere, a mode like a Kelvin wave and a mode like a mixed Rossby–gravity wave both exist, although the parameter which varies continuously in this problem is not the wavenumber (Longuet-Higgins 1968).

As we see from these examples, in rotating shallow water systems, besides the families of Poincaré and Rossby modes, modes which do not completely belong to either family sometimes but not always appear. The aim of this paper is to give a simple criterion to judge whether modes like Kelvin or mixed Rossby–gravity waves (we will call these modes transition modes in this paper) exist. In §2, we will discuss the general theory for such modes, according to the conservation of the number of zeros in an eigenfunction, and by investigating the behaviour of the dispersion curves in large- and small-wavenumber limits. In §3, we will interpret the result in physical terms. After expanding the theory in §4, we will apply it to well-known systems such as edge and continental-shelf waves and equatorial waves.

† The mode of mixed Rossby–gravity wave is so called because its nature changes from that of Rossby waves to that of inertial gravity waves as the wavenumber shifts. There seem to be two ways of looking at this feature. One is that this mode behaves like a Rossby mode at large wavenumbers, and becomes like an inertial gravity wave when the wavenumber is small. The other is that, taking the negative-wavenumber region also in consideration, the nature of this mode varies from that of a Rossby wave to that of a inertial gravity wave, between positive and negative infinite wavenumber limits. In this paper, we will use the former meaning, following the original paper by Matsuno (1966).

2. General theory for transition modes

2.1. *Basic equations and boundary conditions*

We will consider shallow water waves in a channel of width $y_1 < y < y_2$ uniform in the x-direction. The linearized basic equations whose solutions are proportional to $e^{i(kx-\omega t)}$ are

$$-i\omega U = fV - ikgH\eta, \tag{2.1}$$

$$-i\omega V = -fU - gH\,d\eta/dy, \tag{2.2}$$

$$-i\omega\eta = -ikU - dV/dy, \tag{2.3}$$

where $(U, V) \equiv (Hu, Hv)$ is the mass flux, η the vertical displacement of the free surface, f the Coriolis parameter, g the acceleration due to gravity, and H the depth of the fluid. The values of f and H may vary in the y-direction, but we assume $d/dy(f/H) > 0$ and $f > 0$.

We will examine a variety of boundary conditions, as shown in figure 2, in order for this theory to be as general as possible. The boundary conditions considered here are as follows: (*a*) $v = 0$ (so-called rigid boundary); (*b*) $H = 0$ at the boundary, and the variables remain finite (a model of a beach or a continental shelf; since H does not vanish at $y = y_1$ on account of the assumption $d/dy(f/H) > 0$, this boundary condition is applied only to the boundary $y = y_2$); (*c*) $\eta = 0$ (the depth of the fluid suddenly becomes infinite at this boundary and the region of $H = \infty$ extends beyond); (*d*) $H \rightarrow \infty$ (at finite y) and the variables remain finite (from the equation of continuity, η vanishes at this boundary, and this case is almost the same as case (*c*); on account of the assumption, this is applied only to the boundary $y = y_1$); (*e*) $u = iv$ or $k\eta = d\eta/dy$ for $y = y_1$, $u = -iv$ or $k\eta = -d\eta/dy$ for $y = y_2$ (this boundary condition is applied to a boundary beyond which there is a region with a rigid lid, constant depth and constant Coriolis parameter (Orlanski 1968; Iga 1993)); (*f*) $y_1 \rightarrow -\infty (y_2 \rightarrow +\infty)$, and the variables converge to zero (in order for all the modes to be trapped, either f or H must be infinite (Huthnance 1975); owing to the assumption, H is infinite as $y_1 \rightarrow -\infty$, and f is infinite as $y_2 \rightarrow +\infty$).

2.2. *Definition of each mode*

Before solving the governing equations under the boundary conditions given above, we must define how to classify the modes in our problem. In this paper, we define each mode according to the behaviour of the dispersion curves in small- and large-wavenumber limits (figure 3): Rossby modes for which ω vanishes as $k \rightarrow 0$ and ω remains finite (including zero) as $k \rightarrow \infty$; Poincaré modes for which ω remains finite as $k \rightarrow 0$ and $|\omega|$ becomes infinite as $k \rightarrow \infty$; Kelvin modes for which ω vanishes as $k \rightarrow 0$ and $|\omega|$ becomes infinite as $k \rightarrow \infty$; and mixed Rossby–gravity modes for which ω remains finite as $k \rightarrow 0$ and ω remains finite (including zero) as $k \rightarrow \infty$.

This definition is consistent with the nomenclature in the well-known examples given in the introduction. Further, we will call Kelvin and mixed Rossby–gravity modes 'transition modes', because they both shift from the Rossby-wave region to the inertial gravity-wave region with the change of wavenumber, and have intermediate features.

2.3. *Conservation of the number of zeros of U*

By investigating the eigenvalues and the eigenfunctions in limiting cases of $k \rightarrow 0$ and $k \rightarrow \infty$ and connecting them with the aid of some marker, we can identify each mode defined above. In this subsection, we show that the number of zeros pertaining to U of the eigenfunction can be used as such a marker, because they are conserved for each mode even if the wavenumber shifts.

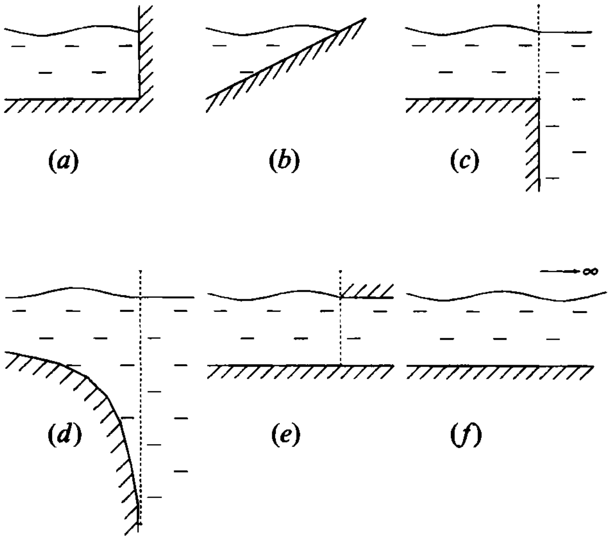

FIGURE 2. Boundary conditions considered: (a) $v = 0$, (b) $H = 0$, (c) $\eta = 0$, (d) $H \to \infty$, (e) $u = \pm \mathrm{i} v$, (f) semi-infinite region.

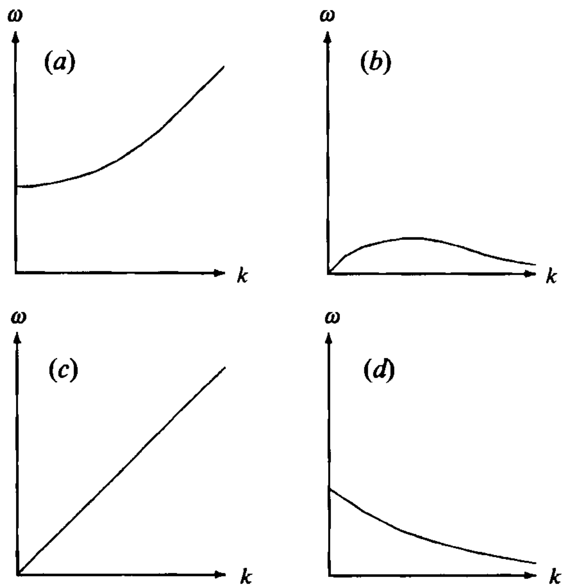

FIGURE 3. Sketches of the dispersion curves: (a) Poincaré mode, (b) Rossby mode, (c) Kelvin mode, (d) Mixed Rossby–gravity mode.

Eliminating U from (2.1), (2.2) and (2.3), we get the differential equations for V and η:

$$\frac{\mathrm{i}(\omega^2 - f^2)}{gH} V = kf\eta + \omega \frac{\mathrm{d}\eta}{\mathrm{d}y}, \tag{2.4}$$

$$(\omega^2 - k^2 gH)\,\eta = \mathrm{i}kfV - \mathrm{i}\omega \frac{\mathrm{d}V}{\mathrm{d}y}. \tag{2.5}$$

From (2.4) and (2.5), V and η (more strictly, V, v, η and $H\eta$) do not simultaneously vanish for the same y. Otherwise, this would lead to $V \equiv 0$, $\eta \equiv 0$, and consequently $U \equiv 0$, which would mean that the eigenvector is a null vector.

In order for the number of zeros of U to change with the shift of k, a zero of U must enter or leave of the region across one of the boundaries. However, this only occurs in two cases: where the boundary condition is $\eta = 0$ at $y = y_1$ and simultaneously $f(y_1) = 0$ holds; and where the boundary condition at $y = y_2$ is $H = 0$. If a zero of U crossed this boundary, U would vanish there and would yield $\mathrm{i}fV + kgH\eta = 0$ from (2.1). Then, we would obtain $\eta = V = 0$ for any boundary condition except for the two cases mentioned above. (The boundary condition $H \to \infty$ requires extra care, but in this case (2.1) and (2.2) yield not only $\eta = 0$ but also $H\eta = 0$, which permits the same result.)

Strictly speaking, pairs of zeros of U may appear or disappear in the interior region, which also alters the number of zeros of U. This is discussed in detail in Appendix A. We discuss the case of the boundary condition $\eta = 0$ at $y = y_1$ and $f(y_1) = 0$, and the case of the boundary condition $H = 0$ at $y = y_2$ in Appendix B. In spite of these exceptions, we can basically state that the number of zeros of U conserve; we have only to modify the discussion a little in such exceptional cases, as shown in Appendices A, B.

2.4. *Equations and boundary conditions in limiting cases*

In this subsection, we will investigate eigenfunctions in limiting cases of $k \to 0$ and $k \to \infty$. First, we will obtain simplified equations in the interior for limiting cases.

(i) $k \to 0$ *and* $\omega \to$ *finite* (*x-direction-uniform limit*)

Neglecting the terms including k in (2.1), (2.2) and (2.3), we get

$$-\mathrm{i}\omega U = fV, \quad -\mathrm{i}\omega V = -fU - gH\frac{\mathrm{d}\eta}{\mathrm{d}y}, \quad -\mathrm{i}\omega\eta = -\frac{\mathrm{d}V}{\mathrm{d}y}.$$

These equations express the one-dimensional motions uniform in the x-direction. Eliminating U and η, we obtain the equation for V:

$$\frac{\mathrm{d}^2 V}{\mathrm{d}y^2} - \frac{f^2}{gH}V + \frac{\omega^2}{gH}V = 0. \tag{2.6}$$

(ii) $k \to 0$ *and* $\omega \to 0$ (*semi-geostrophic limit*)

The terms including k or ω become small in (2.1), (2.2) and (2.3). Moreover, V must become small at the same order, otherwise there would be no other term which balances the term for V in (2.1) or (2.3). Thus, we obtain

$$-\mathrm{i}\omega U = fV - \mathrm{i}kgH\eta, \quad 0 = -fU - gH\frac{\mathrm{d}\eta}{\mathrm{d}y}, \quad -\mathrm{i}\omega\eta = -\mathrm{i}kU - \frac{\mathrm{d}V}{\mathrm{d}y}.$$

These express the semi-geostrophic motions, which implies the complete geostrophic balance in the y-direction. Eliminating U and V, we obtain the equation for η:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{gH}{f^2}\frac{\mathrm{d}\eta}{\mathrm{d}y}\right) - \eta + \frac{k}{\omega}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{gH}{f}\right)\eta = 0. \tag{2.7}$$

(iii) $k \to \infty$ *and* $|\omega| \to \infty$ (*non-rotating limit*)

In (2.1), (2.2) and (2.3), we will retain the terms including k or ω. Furthermore, we will also retain the terms containing the derivative with respect to y, because they become important somewhere in the region. Thus, we get

$$-\mathrm{i}\omega U = -\mathrm{i}kgH\eta, \quad -\mathrm{i}\omega V = -gH\frac{\mathrm{d}\eta}{\mathrm{d}y}, \quad -\mathrm{i}\omega\eta = -\mathrm{i}kU - \frac{\mathrm{d}V}{\mathrm{d}y}.$$

These correspond to the non-rotating case where the effect of the rotation f is neglected. Eliminating U and V, we obtain the equation for η:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(gH\frac{\mathrm{d}\eta}{\mathrm{d}y}\right) - k^2 gH\eta + \omega^2\eta = 0. \tag{2.8}$$

(iv) $k \to \infty$ *and* $\omega \to$ *finite* (*non-divergent limit*)

We will retain the terms with k and the terms containing the derivative with respect to y in (2.1), (2.2) and (2.3). Furthermore, η must be small in order for the term for η to be comparable with other terms in (2.1) and (2.2). Hence, we obtain

$$-\mathrm{i}\omega U = fV - \mathrm{i}kgH\eta, \quad -\mathrm{i}\omega V = -fU - gH\frac{\mathrm{d}\eta}{\mathrm{d}y}, \quad 0 = -\mathrm{i}kU - \frac{\mathrm{d}V}{\mathrm{d}y}.$$

These correspond to the non-divergent case with a rigid lid above. Eliminating U and η, we obtain the equation for V:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{gH}\frac{\mathrm{d}V}{\mathrm{d}y}\right) - \frac{k^2}{gH}V - \frac{k}{\omega}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{f}{gH}\right)V = 0. \tag{2.9}$$

The boundary conditions are also simplified in the limiting cases. We show the simplified boundary conditions together with the conversion formulae from the variable V or η to U in each limiting case in table 1. Now we have only to solve (2.6), (2.7), (2.8) or (2.9) under each combination of boundary conditions as an eigenvalue problem with respet to the eigenvalue of ω^2 or $-1/\omega$. Note, however, that for $\omega = f(y_1)$ (or $-f(y_2)$) which arises from the boundary condition $u = \pm \mathrm{i}v$ in the limiting case $k \to 0$, $\omega \to$ finite, we can always find a function satisfying this condition. Therefore, we must add the eigenvalue of $\omega = f(y_1)$ (or $-f(y_2)$) to the set of eigenvalues obtained from the condition $V = 0$.

The eigenvalue problems to solve in each limiting case have many combinations of boundary conditions; most of them are of Sturm–Liouville type, for which it is well-known that there is a series of a countable infinite number of positive eigenvalues $0 \leqslant \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ and that the eigenfunction $\varphi_n(y)$ corresponding to the eigenvalue λ_n has n zeros in the interior region (e.g. Courant & Hilbert 1931). For the problems considered here, some boundary conditions include eigenvalues in their expressions, and therefore are not precisely Sturm–Liouville problems. Nevertheless, we have only slightly to modify the conclusions on their eigenvalues and the eigenfunctions as follows. The proof is given in Appendix C.

THEOREM. *In the case where the boundary condition at* $y = y_1$ *is* $v = 0$, *in the limit of* $k \to 0$, $\omega \to 0$ *or in the case where the boundary condition at* $y = y_2$ *is* $\eta = 0$, *in the limit of* $k \to \infty$, $\omega \to$ *finite, there is only one negative eigenvalue* $\lambda_{-1} < 0$ *and there are an infinite number of positive eigenvalues* $0 \leqslant \lambda_0 < \lambda_1 < \lambda_2 < \ldots$. *The eigenfunction* $\varphi_{-1}(y)$

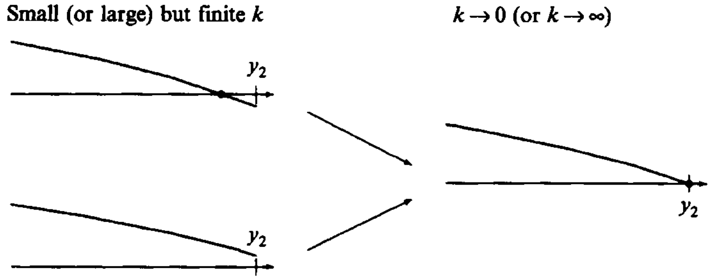

FIGURE 4. Behaviour of zeros of U near a boundary of the region for small (large) but finite k, in the case where a zero of U is located exactly at the boundary when k vanishes (becomes infinity). When k remains finite, a zero of U exists in the interior in some cases and not in others.

Boundary conditions	Domain: $\begin{cases} k\to 0 \\ \omega\to \text{finite}\end{cases}$	$\begin{cases} k\to 0 \\ \omega\to 0\end{cases}$	$\begin{cases} k\to \infty \\ \lvert\omega\rvert\to \infty\end{cases}$	$\begin{cases} k\to \infty \\ \omega\to \text{finite}\end{cases}$
$V,\eta \Rightarrow U$	$U=\mathrm{i}\dfrac{f}{\omega}V$	$U=-\dfrac{gH}{f}\dfrac{\mathrm{d}\eta}{\mathrm{d}y}$	$U=\dfrac{kgH}{\omega}\eta$	$U=\mathrm{i}\dfrac{1}{k}\dfrac{\mathrm{d}V}{\mathrm{d}y}$
$v=0$	$V=0$	$\dfrac{\mathrm{d}\eta}{\mathrm{d}y}=-\dfrac{fk}{\omega}\eta$	$\dfrac{\mathrm{d}\eta}{\mathrm{d}y}=0$	$V=0$
$H=0$	$\begin{cases} v: \text{finite}\to \\ V=0\end{cases}$	$\begin{cases} \eta: \text{finite}\to \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}y}=-\dfrac{fk}{\omega}\eta+\dfrac{f^2}{g}\left(\dfrac{\mathrm{d}H}{\mathrm{d}y}\right)^{-1}\eta\end{cases}$	$\begin{cases} \eta: \text{finite}\to \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}y}=-\dfrac{\omega^2}{g}\left(\dfrac{\mathrm{d}H}{\mathrm{d}y}\right)^{-1}\eta\end{cases}$	$\begin{cases} v: \text{finite}\to \\ V=0\end{cases}$
$\left.\begin{matrix}\eta=0 \\ H\to\infty\end{matrix}\right\}$	$\dfrac{\mathrm{d}V}{\mathrm{d}y}=0$	$\eta=0$	$\eta=0$	$\dfrac{\mathrm{d}V}{\mathrm{d}y}=\dfrac{fk}{\omega}V$
$u=\pm\mathrm{i}v$	$\begin{cases} V=0 \\ \text{or } \omega=f(y_1) \\ (\omega=-f(y_2))\end{cases}$	$\dfrac{\mathrm{d}\eta}{\mathrm{d}y}=\mp k\eta$	$\dfrac{\mathrm{d}\eta}{\mathrm{d}y}=\mp k\eta$	$\dfrac{\mathrm{d}V}{\mathrm{d}y}=\mp kV$
$\left.\begin{matrix}y_1\to-\infty \\ y_2\to+\infty\end{matrix}\right\}$	$V\to 0$	$\eta\to 0$	$\eta\to 0$	$V\to 0$

TABLE 1. Boundary conditions expressed in terms of V or η in each limiting case. Conversion formulae from V or η to U are also shown. The upper signs in the row for boundary condition $u=\pm\mathrm{i}v$ denote the condition applied for $y=y_1$, and the lower signs for $y=y_2$

corresponding to λ_{-1} has no zero in the interior region, and the eigenfunctions $\varphi_n(y)$ corresponding to $\lambda_n (n\geqslant 0)$ have n zeros in the interior region. In other cases, the result is the same as the Sturm–Liouville problem, even if the boundary condition includes the eigenvalue: there is an infinite number of positive eigenvalues $0\leqslant\lambda_0<\lambda_1<\lambda_2<\ldots$ and the eigenfunctions $\varphi_n(y)$ corresponding to λ_n have n zeros in the interior region.

From this theorem, we can count the number of zeros pertaining to V of the eigenfunction in the limiting cases of $k\to 0$, $\omega\to$ finite and $k\to\infty$, $\omega\to$ finite, and the number of zeros pertaining to η in the limiting cases of $k\to 0$, $\omega\to 0$ and $k\to\infty$, $|\omega|\to\infty$.

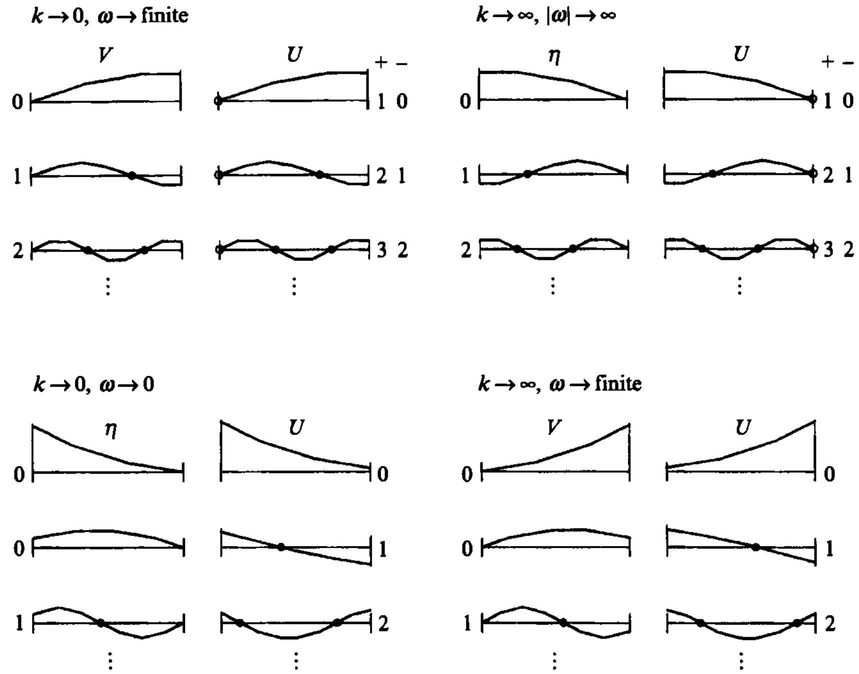

FIGURE 5. Sketches of V or η and U for the three gravest modes in each limiting case. Zeros in the interior region are indicated by solid circles. Zeros of U located exactly at the boundary (indicated by open circles) may enter the interior region or disappear to the exterior when k becomes finite, depending on the sign of ω. Hence, the number of zeros of U for finite k are shown for both signs of ω. Only the case $v = 0$ at $y = y_1$, $\eta = 0$ at $y = y_2$ is shown, but we can easily draw similar figures for other combinations of boundary conditions.

2.5. *The number of each mode*

Since we already know the number of zeros of V or η in each limiting case, we can also count the number of zeros of U, using the conversion formulae from V or η to U in each case. We must, however, take care in some cases: those where a zero of U exists at the boundary in the limiting case. We must ascertain whether the zero of U located exactly at the boundary in the limiting case is originated from the interior or exterior region when the parameter (wavenumber for the present) remains finite (figure 4). A discussion on such zeros of U is given in Appendix D. Considering such behaviour of U near the boundaries, we count the number of zeros of U for each mode. Only the case $v = 0$ at $y = y_1$, $\eta = 0$ at $y = y_2$ is shown in figure 5, but we can count the zeros similarly in other cases.

Now that we know the number of zeros of U in each limiting case, we can connect the limiting cases of large and small wavenumbers using the dispersion curves, on the basis of the conservation of zeros of U. The connections for various boundary conditions are shown in figure 6. Further, the boundary condition $H \to \infty$ is identical to the boundary $\eta = 0$, since the condition $H \to \infty$ leads to the condition $\eta = 0$. The boundary condition $H = 0$ results in the same conclusions as the boundary condition $v = 0$, as shown in Appendix B, and we can easily see that the cases of $y_1 \to -\infty$ and $y_2 \to +\infty$ are the same as the boundary condition $u = \pm iv$. (The boundary condition $y_1 \to -\infty$ or $y_2 \to +\infty$ is expressed as $\eta \to 0$ for the semi-geostrophic limit and the non-rotating limit, and as $V \to 0$ for the x-direction-uniform limit and the non-divergent limit; these are the same as for the boundary condition $m = 0$, which we will discuss

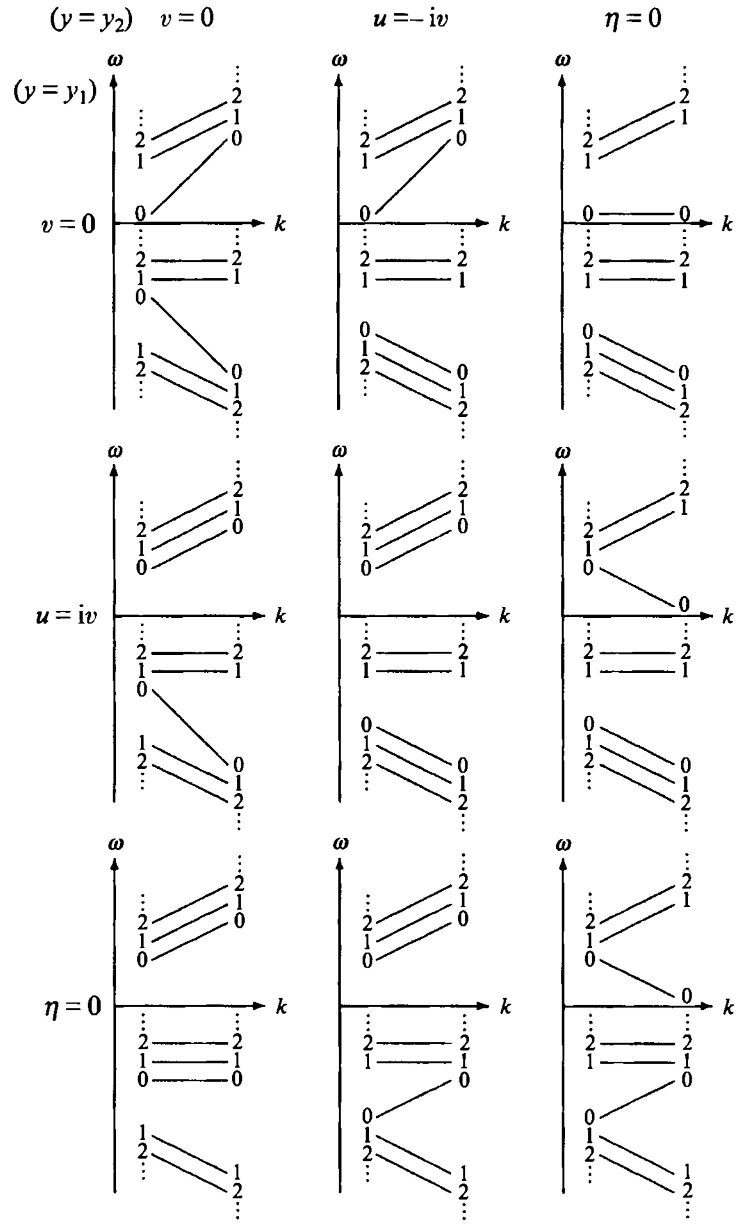

FIGURE 6. Sketches of the dispersion curves for each combination of the boundary conditions. Numbers indicate how many zeros of U exist in the interior region. Modes shifting to a different group between $k \rightarrow 0$ and $k \rightarrow \infty$ are transition modes.

in §4.3.) From figure 6, we see the following features. There are infinite Poincaré modes consisting of the family with $\omega > 0$ and that with $\omega < 0$. There are also infinite Rossby modes, whose frequencies are negative except for the gravest mode in the case where the boundary conditions are $v = 0$ at $y = y_1$ and $\eta = 0$ at $y = y_2$. Aside from these, there may be transition modes, depending on the boundary conditions. In table 2, we show which transition modes exist for each combination of boundary conditions.

	$y = y_2$		
$y = y_1$	$\begin{cases} v = 0 \\ H = 0 \end{cases}$	$\begin{cases} u = -\mathrm{i}v \\ y_2 \to +\infty \end{cases}$	$\eta = 0$
$v = 0$	$\begin{cases} \text{K with } \omega > 0 \\ \text{K with } \omega < 0 \end{cases}$	K with $\omega > 0$	None
$\begin{rcases} u = \mathrm{i}v \\ y_1 \to -\infty \end{rcases}$	K with $\omega < 0$	None	M with $\omega > 0$
$\begin{rcases} \eta = 0 \\ H \to \infty \end{rcases}$	None	M with $\omega < 0$	$\begin{cases} \text{M with } \omega > 0 \\ \text{M with } \omega < 0 \end{cases}$

TABLE 2. Transition modes for each combination of boundary conditions. K indicates a Kelvin mode and M a mixed Rossby–gravity mode

3. Explanation of transition modes in terms of boundary waves

Discussion in the previous section has enabled us to determine mathematically the condition for transition modes to exist, but how should we understand this result in physical terms? In this section, we will explain the result by relating transition modes to boundary waves.

First, we will briefly review the Kelvin waves. A typical Kelvin wave exists in a semi-infinite region of constant depth bounded by a rigid wall, and its velocity component normal to the boundary is identically zero. In other words, if we let f be constant in (2.1), (2.2) and (2.3), and the boundary conditions are

$$v = 0 \quad \text{at} \quad y = 0, \quad \text{variables} \to 0 \quad \text{as} \quad y \to \infty,$$

a mode satisfying $V \equiv 0$ exists:

$$\omega = k(gH)^{1/2}, \quad (U, V, \eta) = ((gH)^{1/2}, 0, 1)\exp\left[-\int^y \mathrm{d}y/\lambda_R(y)\right],$$
$$\text{where} \quad \lambda_R(y) \equiv (gH)^{1/2}/f(y), \quad (3.1)$$

which is typically called a Kelvin wave. This wave decays exponentially away from the boundary $y = y_1$ unless λ_R strongly varies, and has the nature of a boundary wave whose energy is trapped near the boundary. Therefore, if we define the mode with $\omega = k(gH(y_1))^{1/2}$ as the corresponding typical Kelvin wave for the boundary condition $v = 0$ at $y = y_1$, and $\omega = -k(gH(y_2))^{1/2}$ for the boundary condition $v = 0$ at $y = y_2$, we can expect almost the same mode as this corresponding typical Kelvin wave to exist in more general cases with varying depth or with a finite region, as long as the boundary condition $v = 0$ is retained.

However, for the boundary conditioin $H(y_2) = 0$, defining the corresponding typical Kelvin wave in the above manner would cause the frequency to vanish and is therefore inadequate. In this instance, we should regard as the corresponding typical Kelvin wave the mode which appears when $H = (-\mathrm{d}H/\mathrm{d}y\,|_{y=y_2})(y_2 - y), f = f(y_2)$, namely the Kelvin-wave-like mode present in Reid's model for edge and continental-shelf waves:

$$\left.\begin{aligned} \omega &= \tfrac{1}{2}[f(y_2) - (f(y_2)^2 + 4kg\delta)^{1/2}], \\ (U, V, \eta) &= (-\delta(y_2 - y), -\mathrm{i}\delta(y_2 - y), (\omega + f(y_2))/(kg))\,\mathrm{e}^{-k(y_2 - y)}, \\ \text{where} \quad \delta &\equiv -\mathrm{d}H/\mathrm{d}y\,|_{y=y_2}. \end{aligned}\right\} \quad (3.2)$$

A similar argument applies to the boundary condition $\eta = 0$ (or $H \to \infty$). Just as a wave of $V \equiv 0$ exists under the boundary condition $v = 0$, a boundary wave whose

$y = y_1$		
$v = 0$	$\begin{cases} u = \mathrm{i}v \\ y_1 \to -\infty \end{cases}$	$\begin{cases} \eta = 0 \\ H \to \infty \end{cases}$
K with $\omega > 0$	None	I with $\omega < 0$

$y = y_2$		
$\begin{cases} v = 0 \\ H = 0 \end{cases}$	$\begin{cases} u = -\mathrm{i}v \\ y_2 \to +\infty \end{cases}$	$\eta = 0$
K with $\omega < 0$	None	I with $\omega > 0$

TABLE 3. Boundary waves for each boundary condition. K indicates a Kelvin wave and I an inertial oscillation

surface elevation identically vanishes is present in a semi-infinite region with constant Coriolis parameter bounded by an open boundary of $\eta = 0$. That is, if we let f be constant in (2.1), (2.2) and (2.3), and the boundary conditions are

$$\eta = 0 \quad \text{at} \quad y = 0, \quad \text{variables} \to 0 \quad \text{as} \quad y \to \infty,$$

a mode satisfying $\eta \equiv 0$ exists:

$$\omega = -f, \quad (U, V, \eta) = (1, \mathrm{i}, 0)\, \mathrm{e}^{-ky}. \tag{3.3}$$

This wave is commonly known as an inertial oscillation and also decays away from the boundary. Like the Kelvin wave, almost the same mode as the corresponding typical inertial oscillation exists, even if the Coriolis parameter varies or if the region is not semi-infinite, as long as the boundary condition $\eta = 0$ is retained; the corresponding typical inertial oscillation may be defined as the mode with $\omega = -f(y_1)$ for the boundary condition $\eta = 0$ at $y = y_1$, and as the mode with $\omega = f(y_2)$ for the boundary condition $\eta = 0$ at $y = y_2$.

Based on this discussion, we may classify the boundary conditions into three categories (table 3): closed boundaries such as $v = 0$ or $H = 0$ accompanied by a Kelvin wave; open boundaries such as $\eta = 0$ or $H \to \infty$ accompanied by an inertial oscillation; and neutral boundaries such as $u = \pm \mathrm{i}v$ or a semi-infinite region accompanied by no boundary wave. A typical Kelvin wave satisfies the condition for the Kelvin mode defined in §2.2, and a typical inertial oscillation satisfies the condition for the mixed Rossby–gravity mode. Hence, let us regard the Kelvin wave as corresponding to the Kelvin mode and the inertial oscillation as corresponding to the mixed Rossby–gravity mode. Comparing table 2 with table 3 and considering this corresponence, we can understand most of the results in table 2. However, two disagreements still remain. In the cases where one boundary condition is $\eta = 0$ (or $H \to \infty$) and the other is $v = 0$ (or $H = 0$), there is no transition mode, although a Kelvin wave and an inertial oscillation both exist as boundary waves. Figure 7 shows the dispersion relation under the boundary conditions $\eta = 0$ at $y = y_1$ and $v = 0$ at $y = y_2$, when certain functions of f and H are given. This figure shows that, although a mode like a Kelvin wave and a mode like an inertial oscillation both exist, they interchange between the limits of large and small wavenumbers (interchanged modes are sometimes called kissing modes and are a common occurrence), and the transition modes seem to disappear.

Noting such interchanges, we can fully understand the conclusions obtained in the previous section by interpreting as follows. First, consider the boundary waves

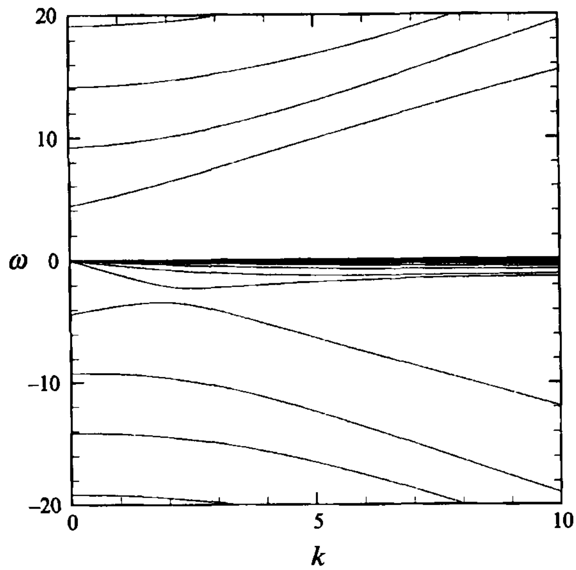

FIGURE 7. Dispersion curves for $f = 1+5y$, $gH = 6-5y$, $\eta = 0$ at $y = 0$ and $v = 0$ at $y = 1$. A Kelvin wave, for which ω vanishes as k vanishes and ω becomes infinite as $k \to \infty$, and an inertial oscillation, for which ω remains finite for all k, are interchanged around $k \sim 2.2$, $\omega \sim -2.5$, and transition modes seem to have disappeared.

corresponding to the boundary conditions. Basically, these boundary waves become transition modes. (A Kelvin wave becomes a Kelvin mode and an inertial oscillation becomes a mixed Rossby–gravity mode.) If there are boundary waves whose dispersion curves overlap, however, the waves interchange on the way between the limits of large and small wavenumbers. (A Kelvin wave or an inertial oscillation may also interchange with inertial gravity waves or Rossby waves, but these interchanges do not affect the number of transition modes.)

4. Extension of the theory

The problem of free oscillations over a sphere is beyond the discussion given in §2. We will extend the theory a little more in order to apply it to this problem as well.

4.1. *Extension to cases with a metric factor*

We discussed the fluid over a plane in §2, but the equations over a sphere include a metric factor, which will require modifying the equations.

If we include the metric factor $m(y)$, the linearized basic equations whose solutions are proportional to $e^{i(kx-\omega t)}$, which correspond to (2.1), (2.2) and (2.3), are

$$-i\omega U = fV - \frac{ikgH}{m}\eta, \tag{4.1}$$

$$-i\omega V = -fU - gH\frac{d\eta}{dy}, \tag{4.2}$$

$$-i\omega\eta = -\frac{ik}{m}U - \frac{1}{m}\frac{d}{dy}(mV). \tag{4.3}$$

The equations in the x-direction-uniform limit (corresponding to (2.6)), in the semi-geostrophic limit (corresponding to (2.7)), in the non-rotating limit (corresponding to (2.8)) and in the non-divergent limit (corresponding to (2.9)) respectively become as follows:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}y}(mV)\right]-\frac{f^2}{gH}V+\frac{\omega^2}{gH}V=0, \tag{4.4}$$

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{mgH}{f^2}\frac{\mathrm{d}\eta}{\mathrm{d}y}\right)-\eta+\frac{k}{m\omega}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{gH}{f}\right)\eta=0, \tag{4.5}$$

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}y}\left(mgH\frac{\mathrm{d}\eta}{\mathrm{d}y}\right)-\frac{k^2gH}{m^2}\eta+\omega^2\eta=0, \tag{4.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{m}{gH}\frac{\mathrm{d}}{\mathrm{d}y}(mV)\right]-\frac{k^2}{gH}V-\frac{k}{\omega}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{f}{gH}\right)mV=0. \tag{4.7}$$

Since these equations have self-adjoint forms with respect to η or mV, we have only to replace V by mV to discuss them in the same way as earlier, and we obtain the same conclusion on the number of zeros pertaining to U of the nth mode for each boundary condition.

4.2. *Limiting cases with respect to a parameter other than k*

In §2, we discussed the limiting cases with respect to the wavenumber k. Here we will examine the behaviour of the modes when a parameter other than the wavenumber is shifted. If we non-dimensionalize the variables by characteristic values, (4.1), (4.2) and (4.3) become equations characterized by just one non-dimensional parameter $\epsilon \equiv f_*^2 L_*^2/g_* H_*$ (Longuet-Higgins 1968). The shifts of a parameter other than wavenumber are reduced to that of ϵ. (Here, f_*, g_*, H_* and L_* express the characteristic values of f, g, H and the horizontal scale used in the non-dimensionalization, respectively.) The limiting cases lead to the same simplified equations leading up to (2.6)–(2.9) (or (4.4)–(4.7)), so that there is a correspondence between the limits on k and $1/\epsilon$. In this discussion, we assume that H, f and m do not vanish but remain finite at any boundary. Even if one of them vanishes, however, we can classify the limiting cases in the same way, based on the criterion $\omega \lesseqgtr O(\epsilon^{\alpha})$ using a certain value of α, instead of classifying them by $\omega \to 0$, finite or ∞.

4.3. *Boundary condition of* $m = 0$

When the metric factor is included, besides the boundary conditions mentioned earlier, we can consider also the boundary condition that the metric vanishes but that the variables remain finite at this boundary. We will briefly examine the features of this boundary condition, which we will call the boundary condition $m = 0$ hereinafter. The condition $m = 0$ leads to $\eta = 0$ and $mV = 0$. (Physically $m = 0$ is a pole of the coordinate system and when H is not singular, a scalar variable must vanish unless $k = 0$, and a vector variable must vanish unless $k = \mathrm{d}m/\mathrm{d}y\,|_{y=y_1}$ (or $k = \mathrm{d}m/\mathrm{d}y\,|_{y=y_2}$).) The boundary condition $m = 0$ results in the same conclusions for the condition $\eta = 0$ for the semi-geostrophic limit and the non-rotating limit where the equations are expressed by η, and in almost the same conclusions for $v = 0$ for the x-direction-uniform limit and the non-divergent limit where the equations are expressed by mV. However, a zero of U located just at the boundary in the limiting cases does not shift but remains there even if the parameter becomes finite. Therefore, although we had to add a zero of U located near the boundary when the parameter was finite in §2.5, we need not here. Considering these features, we can easily see that the boundary condition $m = 0$ is a neutral boundary like the boundary condition $u = \pm iv$.

5. Application of the theory to well-known examples

In this section, we will apply the theory discussed in earlier sections to well-known examples.

(i) *β-channel between two rigid walls*

This problem is often considered in textbooks to explain Rossby modes and Poincaré modes (e.g. Pedlosky 1987). Since the boundary conditions are both $v = 0$, there are two Kelvin waves which propagate in both directions.

(ii) *Various kinds of edge and continental-shelf waves*

We will also consider models of edge and continental-shelf waves mentioned in the introduction. Since one of the boundary conditions is $H = 0$ in all cases, a Kelvin wave accompanies this boundary. In the model of Reid (1958) the other boundary condition is $y_1 \to -\infty$, and in that of Iga (1993) it is $u = iv$; they are both neutral boundaries. Hence only one Kelvin mode exists as a transition mode in these two cases. On the other hand, in Mysak's (1968) model, since the other boundary condition is an open boundary of $\eta = 0$, an inertial oscillation accompanying this boundary also exists. The Kelvin wave and the inertial oscillation interchange and there is no transition mode.

(iii) *Equatorial waves*

Next we will consider the equatorial waves or the normal modes of shallow water waves over an equatorial β-plane with constant depth, investigated by Matsuno (1966). This situation does not satisfy $f > 0$ and we cannot directly apply the theory to this problem. If we separate the modes into symmetric and antisymmetric modes, however, the problem for the symmetric (antisymmetric) modes is equivalent to that in the region $y > 0$ bounded by the boundary condition $v = 0$ ($\eta = 0$) at $y = 0$. We can apply the theory to these separated problems. Since the other boundary condition at $y = y_2$ is $y_2 \to \infty$ for both problems, a Kelvin mode exists for the symmetric modes, and a mixed Rossby–gravity mode for the antisymmetric modes. As for the equatorial Kelvin wave, it is often stated that the equator plays a role of a rigid wall, causing the Kelvin wave to exist. If we follow the explanation, we can state that the equator also plays the role of an open boundary, causing the mixed Rossby–gravity wave to exist as a boundary wave.

It is suggested that the mixed Rossby–gravity mode defined in this paper is essentially a modification of an inertial oscillation, since we can find the corresponding typical inertial oscillation. For this equatorial so-called mixed Rossby–gravity mode, however, it is difficult to find such a corresponding typical inertial oscillation, because f vanishes at the boundary $\eta = 0$. We should regard this mixed Rossby–gravity mode, which is obtained in the situation $f = \mathrm{d}f/\mathrm{d}y\,|_{y=y_1}(y-y_1)$:

$$\omega = \tfrac{1}{2}[k(gH)^{1/2} - (k^2 gH + 4(gH)^{1/2}\beta)^{1/2}],$$

$$(U, V, \eta) = (\beta(y-y_1), \mathrm{i}(k(gH)^{1/2} - \omega), \beta/(gH)^{1/2}(y-y_1))\,\mathrm{e}^{(y-y_1)^2/2\lambda_E^2}, \qquad (5.1)$$

where $\lambda_E^2 \equiv (gH)^{1/2}/\beta, \quad \beta \equiv \mathrm{d}f/\mathrm{d}y\,|_{y=y_1},$

as the typical inertial oscillation for the cases $\eta = 0$ at $y = y_1$ and $f(y_1) = 0$, in the same way as we considered the Kelvin mode present in the situation

$$H = (-\mathrm{d}H/\mathrm{d}y|_{y=y_2})(y_2 - y)$$

as the typical Kelvin wave for the case $H = 0$ at $y = y_2$.

(iv) *Free oscillations on a rotating sphere*

With the extended theory in §4, we can also apply our theory to the free oscillations on a sphere investigated by Longuet-Higgins (1968). We will separate this problem into those for symmetric and antisymmetric modes, as we did for the equatorial waves. The boundary conditions for the symmetric modes are $v = 0$ and $m = 0$, and for the antisymmetric modes $\eta = 0$ and $m = 0$. Since the boundary conditions $v = 0$, $\eta = 0$ and $m = 0$ are closed, open and neutral boundaries, respectively, a Kelvin mode and a mixed Rossby–gravity mode appear from the symmetric and antisymmetric problems, respectively.

6. Conclusions

We have investigated generally the modes of shallow water waves in a channel wherein the Coriolis parameter f and the depth H vary in the y-direction (but f and $(\mathrm{d}/\mathrm{d}y)(f/H)$ are positive). As a result, we have proved mathematically the following features.

(i) In rotating shallow water system, there always exist families of Poincaré modes propagating in the positive and negative directions, and a family of Rossby modes whose phase propagates slowly in the negative direction.

(ii) Under certain boundary conditions, there sometimes also exist Kelvin modes or mixed Rossby–gravity modes whose features vary as the wavenumber (or another parameter) shifts.

We can interpret these results as follows. In a rotating shallow water system, there exist inertial gravity waves and Rossby waves. Moreover, if a boundary condition is a closed boundary, a Kelvin wave accompanying this boundary appears, and if a boundary condition is an open boundary, an inertial oscillation appears. Rossby waves, inertial gravity waves, Kelvin waves and inertial oscillations correspond in the actual normal modes to Rossby modes, Poincaré modes, Kelvin modes and mixed Rossby–gravity modes, respectively. Nevertheless, when two dispersion curves cross, they interchange in the actual dispersion relation. In particular, if a Kelvin wave and an inertial oscillation interchange, the transition modes disappear. By applying this theory, we can systematically understand Kelvin modes and mixed Rossby–gravity modes in well-known problems.

The author thanks Professor R. Kimura for his encouragement throughout this study. The IMSL Library was used to solve the eigenvalue problem and the NCARG Library to draw figure 7.

Appendix A. Zeros of U which appear or disappear in pairs

When we discussed the conservation of zeros of U (which we call N_U hereinafter) in §2.3, we did not completely deny the possibility of pairs of zeros of U appearing or disappearing in the interior region. When that happens, U and $\mathrm{d}U/\mathrm{d}y$ simultaneously vanish there. From (2.1), (2.2) and (2.3), we obtain a relation to express $\mathrm{d}U/\mathrm{d}y$ in terms of V and η:

$$\frac{\mathrm{d}U}{\mathrm{d}y} = \mathrm{i}\left(\frac{1}{\omega}\frac{\mathrm{d}f}{\mathrm{d}y}+k\right)V+\left(\frac{kg}{\omega}\frac{\mathrm{d}H}{\mathrm{d}y}-f\right)\eta. \tag{A 1}$$

Using (2.1) and (A 1), $U = \mathrm{d}U/\mathrm{d}y = 0$ would usually lead to $V = \eta = 0$, which implies a null eigenvector. Therefore, such an appearance and disappearance of a pair of zeros usually does not occur. However, where the determinant of the coefficient matrix $(1/\omega)\,\mathrm{d}(f/H)/\mathrm{d}y+k/H+f^2/kgH^2$ vanishes, V and η do not necessarily vanish. Hence the number of zeros of U is not strictly conserved, and the discussion in the text would lose its validity. Then what kind of modification should we add in order to make the discussion exact?

Basically, the zeros of U and the zeros of V align alternately for the following reason. Eliminating η from (2.1) and (2.3), we find $(kgH\mathrm{i})\,\mathrm{d}V/\mathrm{d}y = (k^2gH-\omega^2)\,U+\omega f\mathrm{i}V$. From this relation, we can see that $(k^2gH-\omega^2)\,U$ has opposite sign at the points where a certain zero of V exists and where the neighbouring zero exists. Particularly, unless $k^2gH-\omega^2$ vanishes, a zero of U always exists between two neighbouring zeros of V. Similarly, unless ω^2-f^2 vanishes, a zero of U always exists between two neighbouring zeros of η, and unless $(1/\omega)\,\mathrm{d}(f/H)/\mathrm{d}y+k/H+f^2/kgH^2$ vanishes, a zero of V always exists and, as does a zero of η between two neighbouring zeros of U.

If a pair of zeros of U appears in the interior, three zeros of U align in succession. To exclude such additional zeros of U, we count the zeros of U as follows: If zeros of U align in succession, or zeros of V do, continue to remove these zeros by pairs. When the zeros of U and V align alternately after repeating this operation, we count the number of remaining zeros of U, which we will call $N_{U,V}$ hereinafter. Even if wavenumber k shifts, a zero of U never change places with a zero of V, otherwise U and V would simultaneously vanish when they are replaced, which would lead to a null eigenvector. Further, even if zeros of U or V appear or disappear in the interior region, they always do so in pairs. Therefore, the $N_{U,V}$ never alter, and are conserved strictly. If we define $N_{U,\eta}$ in the same way, the $N_{U,\eta}$ are also conserved.

We counted the numbers of zeros of U in limiting cases in §2.5, but we must ascertain which of N_U, $N_{U,V}$ and $N_{U,\eta}$ is strictly equal to what we counted in each limiting case.

(i) *The limit $k\to 0$, $\omega\to$ finite*

Obviously, here we counted N_U. Since $\eta = (-1/f)\,\mathrm{d}U/\mathrm{d}y$, an odd number of zeros of η exists between two neighbouring zeros of U, and hence N_U equals $U_{U,\eta}$. Moreover, zeros of V located exactly where zeros of U exist when $k = 0$ move a little from the zeros of U if k becomes small but finite. While k is small enough, $k^2gH-\omega^2$ never vanishes in the region, and all the zeros of V move in the same direction. Therefore N_U also equals $N_{U,V}$.

(ii) *The limit $k\to 0$, $\omega\to 0$*

Since $U=(-gH/f)\,\mathrm{d}\eta/\mathrm{d}y$, an odd number of zeros of U exists between two neighbouring zeros of η. Hence what we counted is $N_{U,\eta}$. Further, it equals also $N_{U,V}$, because $-\mathrm{i}V$ and η have the same sign where U vanishes (since the relation $\mathrm{i}V = (-kgH/f)\,\eta$ holds there).

(iii) *The limit $k\to\infty$, $|\omega|\to\infty$*

Considering that ω^2-f^2 never vanishes in the region, we can see, in the same way as the limit $k\to 0$, $\omega\to$ finite, that we counted N_U, which is also equal to $N_{U,V}$ and $N_{U,\eta}$.

(iv) *The limit* $k \to \infty$, $\omega \to$ *finite*

In the same way as the limit $k \to 0$, $\omega \to 0$, what we counted equals $N_{U,V}$, and also $N_{U,\eta}$.

We can see from these consideration that what we counted in each limiting case equals both $N_{U,V}$ and $N_{U,\eta}$. Further, as mentioned above, both $N_{U,V}$ and $N_{U,\eta}$ are strictly conserved. Therefore, if we use $N_{U,V}$ or $N_{U,\eta}$ instead of N_U, the discussion in the text becomes exact.

Appendix B. Cases where the number of zeros of U alters

A zero of U may enter or leave across the boundaries in two exceptional cases: the case where the boundary condition at $y = y_1$ is $\eta = 0$ and simultaneously $f(y_1) = 0$ holds; and the case where the boundary condition at $y = y_2$ is $H = 0$. We will examine these cases and show that the number of transition modes under these boundary conditions is the same as that under the boundary conditions $\eta = 0$ (and $f(y_1) \neq 0$) and $v = 0$, respectively.

First, we will consider the case where the boundary condition at $y = y_1$ is $\eta = 0$ and f happens to vanish there. Since $U = 0$ always holds at this boundary from (2.1), $\mathrm{d}U/\mathrm{d}y$ also vanishes when a zero of U crosses this boundary. If $k\omega \neq -\mathrm{d}f/\mathrm{d}y\,|_{y=y_1}$, this does not occur; otherwise, using (A 1), it would lead $V = 0$ and the eigenvector would be a null vector. Since $\mathrm{d}f/\mathrm{d}y\,|_{y=y_1} > 0$, zeros of U do not cross the boundary for the modes with $\omega > 0$ and all the results are the same as those for $\eta = 0$ (and $f(y_1) \neq 0$). Nevertheless, for the modes with $\omega < 0$, a zero of U enters or leaves across the boundary at $k\omega = -\mathrm{d}f/\mathrm{d}y\,|_{y=y_1}$. In other words, when a dispersion curve crosses $k\omega = -\mathrm{d}f/dy\,|_{y=y_1}$, the number of zeros of U either increases or decreases by one. (Zeros of V and η never cross this boundary. Hence, even if a dispersion curve crosses the curve $k\omega = -\mathrm{d}f/\mathrm{d}y\,|_{y=y_1}$ many times, $N_{U,V}$ and $N_{U,\eta}$ do not change if it crosses an even number of times and they increase or decrease by one if it crosses an odd number of times. Whether they increase or decrease depends on whether a zero of V and η exists closer to the boundary than any zero of U.)

Moreover, we should examine the number of zeros of U in each limiting case under the boundary condition $f(y_1) = 0$, $\eta = 0$, in comparison with the boundary condition $f(y_1) \neq 0$, $\eta = 0$. The result is as follows:

(i) *The limit* $k \to 0$, $\omega \to$ *finite*

The number of zeros of U in the case $f(y_1) = 0$ is the same as that in the case $f(y_1) \neq 0$.

(ii) *The limit* $k \to 0$, $\omega \to 0$

The number of zeros of U in the case $f(y_1) = 0$ is the same as that in the case $f(y_1) \neq 0$.

(iii) *The limit* $k \to \infty$, $|\omega| \to \infty$

In the case $f(y_1) = 0$, the zero of U located at $y = y_1$ in the limit $k \to \infty$ remains exactly at $y = y_1$ even when k becomes finite. Hence, although we added a zero approaching the boundary when we counted the zeros of U for finite k, we need not add such a zero now, and the number of zeros of U is less by one than in the case $f(y_1) \neq 0$. There is a zero of V closer to $y = y_1$ than any zero of U, and further, if we consider a small but finite $1/k$, there is also a zero of η closer to $y = y_1$ than any zero of U. Therefore, when a dispersion curve starting here crosses the curve $k\omega = -\mathrm{d}f/\mathrm{d}y\,|_{y=y_1}$, both $N_{U,V}$ and $N_{U,\eta}$ increase by one.

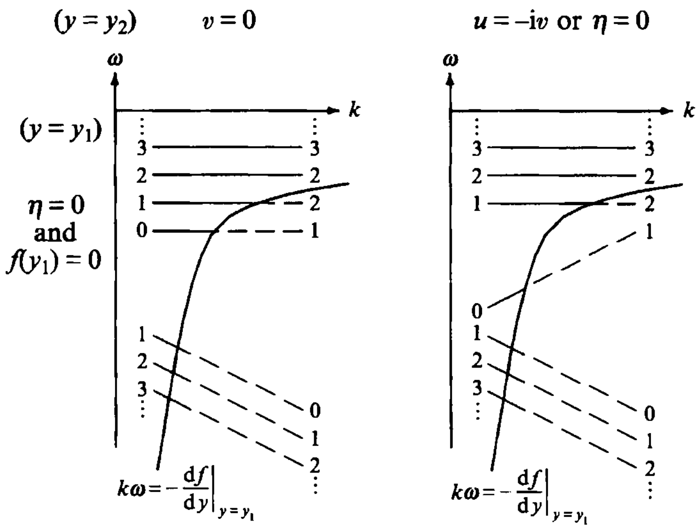

FIGURE 8. Sketches of the dispersion curves of modes with $\omega < 0$ in the case $\eta = 0$ at $y = y_1$ and $f(y_1) = 0$. Numbers indicate how many zeros of U exist in the interior region. In this figure, we show the case where $k\omega_n < -\mathrm{d}f/\mathrm{d}y\,|_{y=y_1}$ holds for $n = 0, 1$ in the limit $k \to \infty$, $\omega \to$ finite, but even when more modes, less modes or even no mode satisfy $k\omega_n < -\mathrm{d}f/\mathrm{d}y\,|_{y=y_1}$, we can draw a figure in the same way. In any case, the existence of transition modes is the same as for the case $\eta = 0$ at $y = y_1$ in figure 6.

(iv) *The limit* $k \to \infty$, $\omega \to$ *finite*

At $y = y_1$, $\mathrm{d}V/\mathrm{d}y$ exactly vanishes. Since $\mathrm{d}^2V/\mathrm{d}y^2 = (k/\omega gH)(\omega k + \mathrm{d}f/\mathrm{d}y)\,V$ holds there, there are the same number of zeros of U as in the case $f(y_1) \neq 0$ if the relation $k\omega > -\mathrm{d}f/\mathrm{d}y\,|_{y=y_1}$ holds, and one more zero than in the case $f(y_1) \neq 0$ if $k\omega < -\mathrm{d}f/\mathrm{d}y\,|_{y=y_1}$. In the latter case, this additional zero of U is located closer to $y = y_1$ than any zero of V and η. Thus, when a dispersion curve starting here crosses $k\omega = -\mathrm{d}f/\mathrm{d}y\,|_{y=y_1}$, $N_{U,V}$ and $N_{U,\eta}$ decrease by one.

From these considerations, the connection between the limits of large and small wavenumbers from the dispersion curves of the modes with $\omega < 0$ under the boundary condition of $\eta = 0$ ann $f(y_1) = 0$ at $y = y_1$, is modified to that shown in figure 8. The number of transition modes is, however, the same as that under the boundary condition $\eta = 0$ and $f(y_1) \neq 0$.

As for the boundary condition $H = 0$ at $y = y_2$, using (A 1) and $\mathrm{d}H/\mathrm{d}y\,|_{y=y_2} < 0$, zeros of U do not cross the boundary for the modes with $\omega > 0$. For the modes with $\omega < 0$, however, when a dispersion curve crosses $f\omega = kg\,\mathrm{d}H/\mathrm{d}y\,|_{y=y_2}$, the number of zeros of U either increases or decreases by one. Under the boundary condition $H = 0$ in comparison with the boundary condition $v = 0$, the number of zeros of U in each limiting case becomes as follows.

(i) *The limit* $k \to 0$, $\omega \to$ *finite*

The zero of U located at $y = y_2$ in the limit of $k \to 0$ remains exactly at $y = y_2$ even when k becomes finite. Hence, although we added a zero approaching the boundary when we count the zeros of U in the interior for finite k, we need not add such a zero now, and the number of zeros of U is less by one than in the case $v = 0$. There is a zero of η closer to $y = y_2$ than any zero of U, and further, if we consider a small but finite

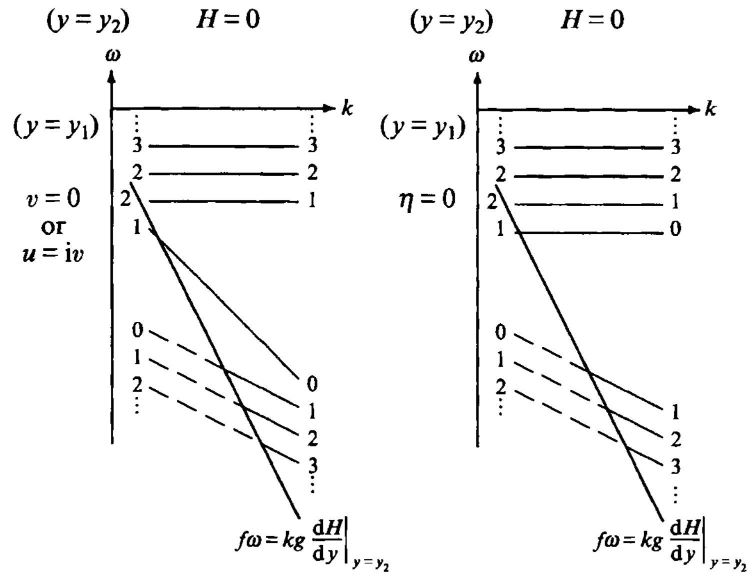

FIGURE 9. Sketches of the dispersion curves of modes with $\omega<0$ in the case $H=0$ at $y=y_2$. Numbers indicate how many zeros of U exist in the interior region. In this figure, we show the case where $f\omega_n < kg\,\mathrm{d}H/\mathrm{d}y|_{y=y_2}$ holds for $n=0,1$ in the limit $k\to 0$, $\omega\to 0$, but even when more modes, less modes or even no mode satisfy $f\omega_n < kg\,\mathrm{d}H/\mathrm{d}y|_{y=y_2}$, we can draw a figure in the same way. In any case, the existence of transition modes is the same as for the case $v=0$ at $y=y_2$ in figure 6.

k, there is also a zero of V closer to $y=y_2$ than any zero of U. Therefore, when a dispersion curve starting here crosses the curve $f\omega=kg\,\mathrm{d}H/\mathrm{d}y|_{y=y_2}$, both $N_{U,V}$ and $N_{U,\eta}$ increase by one.

(ii) *The limit* $k\to 0$, $\omega\to 0$

Since $\mathrm{d}\eta/\mathrm{d}y=(-fk/\omega)\eta+(f^2/g)(\mathrm{d}H/\mathrm{d}y)^{-1}$ holds at $y=y_2$, there is the same number of zeros of U as in the case of $v=0$ if $f\omega > kg\,\mathrm{d}H/\mathrm{d}y|_{y=y_2}$, and there is one more zero than in the case of $v=0$ if $f\omega < kg\,\mathrm{d}H/\mathrm{d}y|_{y=y_2}$. In the latter case, this additional zero of U is located closer to $y=y_2$ than any zero of η and V. Thus, when a dispersion curve starting here crosses the curve $f\omega=kg\,\mathrm{d}H/\mathrm{d}y|_{y=y_2}$, both $N_{U,V}$ and $N_{U,\eta}$ decrease by one.

(iii) *The limit* $k\to\infty$, $|\omega|\to\infty$

The number of zeros of U in the case $H=0$ is the same as that in the case $v=0$. Since $\omega\sim O(k^{1/2})$ as $k\to\infty$, $|f\omega|$ becomes smaller than $|kg\,\mathrm{d}H/\mathrm{d}y|_{y=y_2}|$ for large enough k.

(iv) *The limit* $k\to\infty$, $\omega\to$ *finite*

The number of zeros of U in the case of $H=0$ is the same as that in the case of $v=0$.

The connection between the limits of large and small wavenumbers from the dispersion curves of the modes with $\omega<0$ under the boundary condition $H=0$ at $y=y_2$, is modified to that shown in figure 9. The number of transition modes is, however, the same as that under the boundary condition $v=0$.

Appendix C. The number of zeros of the eigenfunction for extended Sturm–Liouville problems

We will prove the theorem in §2.4 on the number of zeros in the eigenfunction for the extended Sturm–Liouville problem. All of the differential equations in the interior region for the eigenvalue problems which we consider now have the form

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(p(y)\frac{\mathrm{d}\varphi(y)}{\mathrm{d}y}\right)-q(y)\varphi(y)+\lambda\rho(y)\varphi(y)=0, \tag{C 1}$$

where $\quad p(y)\geqslant 0,\quad \rho(y)\geqslant 0,\quad q(y)\geqslant 0.$

The boundary conditions at $y=y_n$ $(n=1,2)$ are

$$\cos\theta_n\frac{\mathrm{d}\varphi}{\mathrm{d}y}=\sin\theta_n\varphi+\lambda h_n\cos\theta_n\varphi, \tag{C 2}$$

where

$$\begin{aligned}&0\leqslant\theta_1\leqslant\tfrac{1}{2}\pi,\quad -\tfrac{1}{2}\pi\leqslant\theta_2\leqslant 0,\\ &h_1>0\quad(\text{when } v=0 \text{ in the limit } k\to 0,\omega\to 0),\\ &h_1\leqslant 0\quad(\text{otherwise}),\\ &h_2<0\quad(\text{when } \eta=0 \text{ in the limit } k\to\infty,\omega\to\text{finite}),\\ &h_2\geqslant 0\quad(\text{otherwise}).\end{aligned}$$

For the boundary condition at $y=y_1$, we will consider here cases other than $v=0$ in the limit $k\to 0$, $\omega\to 0$. In the case $v=0$ in the limit $k\to 0$, $\omega\to 0$, we have only to replace y_1 and y_2 to discuss it in the same way.

First, we will consider the function $\varphi(y,\lambda)$, the solution satisfying the boundary condition at $y=y_1$ and the differential equation with a parameter λ, but not necessarily satisfying the boundary condition at $y=y_2$:

$$\frac{\partial}{\partial y}\left(p(y)\frac{\partial\varphi(y,\lambda)}{\partial y}\right)-q(y)\varphi(y,\lambda)+\lambda\rho(y)\varphi(y,\lambda)=0, \tag{C 3}$$

$$\cos\theta_1\frac{\partial\varphi(y,\lambda)}{\partial y}=\sin\theta_1\varphi(y,\lambda)+\lambda h_1\cos\theta_1\varphi(y,\lambda), \tag{C 4}$$

where $0\leqslant\theta_1\leqslant\frac{1}{2}\pi$, $h\leqslant 0$. We can calculate the value of $[(\partial\varphi(y,\lambda)/\partial y)/\varphi(y,\lambda)]_{y=y_2}$ from this function φ. If this value is equal to $\tan\theta_2+\lambda h_2$, the parameter λ is an eigenvalue of this problem.

Let us define a function $f(y,\lambda)$ as $f(y,\lambda)\equiv\tan^{-1}[p(\partial\varphi/\partial y)/\varphi]$, where we will choose the phase so that the function $f(y,\lambda)$ becomes continuous also across the zeros of φ, and satisfies $-\frac{1}{2}\pi<f(y_1,\lambda)\leqslant\frac{1}{2}\pi$ (figure 10). Then, the function $f(\lambda)$ defined as $f(\lambda)\equiv f(y_2,\lambda)=\tan^{-1}[p(\partial\varphi/\partial y)/\varphi]_{y=y_2}$ behaves as follows.

LEMMA 1. *$f(\lambda)$ is a monotonically decreasing function.*

Proof. Integrating (C 3)$|_{\lambda=\lambda_1}\times\varphi(y,\lambda_2)-$(C 3)$|_{\lambda=\lambda_2}\times\varphi(y,\lambda_1)$ from $y=y_1$ to y_2, using the boundary condition at $y=y_1$ and letting $\lambda_2\to\lambda_1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{p(y_2)\,\partial\varphi(y,\lambda)/\partial y|_{y=y_2}}{\varphi(y_2,\lambda)}\right]=-\frac{1}{\varphi^2(y_2,\lambda)}\left[-h_1p(y_1)\varphi^2(y_1,\lambda)+\int_{y_1}^{y_2}\rho\varphi^2(y,\lambda)\,\mathrm{d}y\right]. \tag{C 5}$$

Since the right-hand side is negative, $[p(\partial\varphi/\partial y)/\varphi]_{y=y_2}$ is a decreasing function of λ

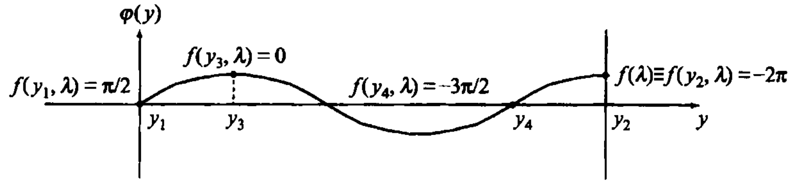

FIGURE 10. Values of functions $f(y,\lambda)$ and $f(\lambda)$ when the function $\varphi(y)$ is given.

except at the points where $\varphi(y_2,\lambda)$ vanishes. Moreover, since we have chosen the phase so that $f(\lambda)$ becomes a continuous function even if $\varphi(y_2,\lambda)$ passes zero, $f(\lambda)$ is a monotonically decreasing function with respect to λ.

LEMMA 2. $f(\lambda)\to-\infty$ as $\lambda\to+\infty$.

Proof. By transformation $\psi\equiv(p\rho)^{1/4}\varphi$, $\xi\equiv\int^{y}(\rho/p)^{1/2}\,\mathrm{d}y$, equation (C 1) becomes

$$\partial^2\psi/\partial\xi^2-r\psi+\lambda\psi=0,$$

where $r\equiv(\mathrm{d}/\mathrm{d}\xi)((p\rho)^{1/4})/(p\rho)^{1/4}+q/\rho$. By this transformation, the interval $[y_1,y_2]$ is transformed to $[\xi_1,\xi_2]$ ($\xi_n\equiv\int^{y_n}(\rho/p)^{1/2}\,\mathrm{d}y$), and zeros of φ in the interval $[y_1,y_2]$ to zeros of ψ in the interval $[\xi_1,\xi_2]$. Since in the limit $\lambda\to+\infty$ the equation is approximated as

$$\partial^2\psi/\partial\xi^2+\lambda\psi\sim 0,$$

the solution ψ becomes

$$\psi\sim\sin(\lambda^{1/2}\xi+\alpha),$$

and the number of zeros of ψ in the interval $[\xi_1,\xi_2]$ increases infinitely with $\lambda\to+\infty$, and so does the number of zeros of φ in the interval $[y_1,y_2]$, which means $f(\lambda)\to-\infty$.

LEMMA 3. $0<f(0)<\tfrac{1}{2}\pi$ *holds*

Proof. We can prove this by showing that φ has no zero in the region $y_1<y<y_2$ and satisfies $p(\partial\varphi/\partial y)/\varphi|_{y=y_2}>0$. Since the boundary condition at $y=y_1$ becomes $\cos\theta_1(\partial\varphi/\partial y)=\sin\theta_1\varphi$ for $\lambda=0$, φ satisfies either

$$\partial\varphi/\partial y|_{y=y_1}\geqslant 0,\ \varphi(y_1)\geqslant 0 \text{ or } \partial\varphi/\partial y|_{y=y_1}\leqslant 0,\ \varphi(y_1)\leqslant 0,$$

and we do not lose the generality by assuming the former case. Suppose that φ had zeros in the interval $y_1<y<y_2$. Then, since φ is positive in $y_1<y<y_3$ (where y_3 is the zero closest to y_1), we would find

$$\frac{\partial}{\partial y}\left(p\frac{\partial\varphi}{\partial y}\right)=q\varphi>0.$$

This inequality and conditions $\partial\varphi/\partial y|_{y=y_1}\geqslant 0$, $\varphi(y_1)\geqslant 0$ would lead to $\varphi|_{y=y_3}>0$, which is inconsistent with the assumption. Hence, φ has no zero in the interval $y_1<y<y_2$. Since φ remains positive throughout this interval, so does $p\,\mathrm{d}\varphi/\mathrm{d}y$; these also hold at $y=y_2$.

LEMMA 4. $p(\partial\varphi/\partial y)/\varphi|_{y=y_2}\sim O((-\lambda)^{1/2})$ *and* $f(\lambda)\to\tfrac{1}{2}\pi$ *as* $\lambda\to-\infty$.

Proof. We can show that φ has no zero if $\lambda<0$ in the same way as in lemma 3. Therefore, if we show that $p(\partial\varphi/\partial y)/\varphi|_{y=y_2}\sim O((-\lambda)^{1/2})$, this implies also $f(\lambda)\to\tfrac{1}{2}\pi$. By the same transformation as in lemma 2, we find

$$\psi\sim e^{(-\lambda)^{1/2}\xi}.$$

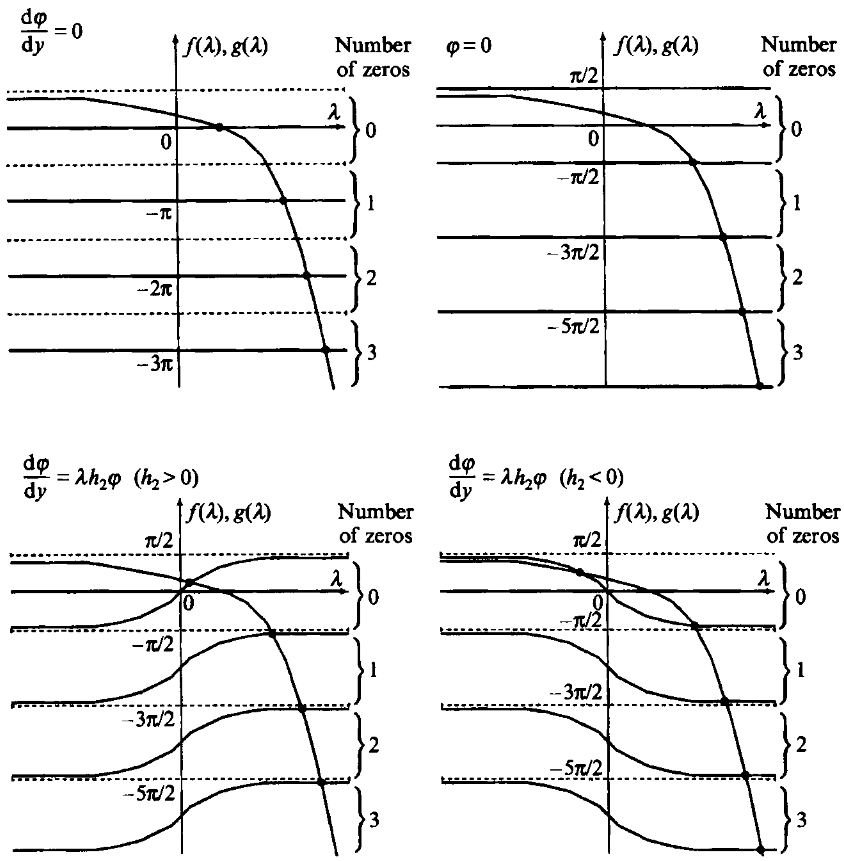

FIGURE 11. Graphs of $f(\lambda)$ and $g(\lambda) \equiv \tan^{-1}[p(y_2)(\sin\theta_2+\lambda h_2)/\cos\theta_2]$ for various boundary conditions at $y = y_2$. The value of λ at the point where they intersect gives the eigenvalue, and we can see the number of zeros from the value of $f(\lambda)$ there. (If $-(n+\frac{1}{2})\pi \leqslant f(\lambda) < -(n-\frac{1}{2})\pi$ holds, there exist n zeros.)

Hence, we get

$$\frac{p\,\partial\varphi/\partial y}{\varphi} \sim (p\rho)^{1/2}(-\lambda)^{1/2} \sim O((-\lambda)^{1/2}).$$

This is valid also at $y = y_2$.

Comparing the graph of $f(\lambda)$ with that of $g(\lambda) \equiv \tan^{-1}[p(y_2)(\tan\theta_2+\lambda h_2)]$ (keeping in mind that $p(y_2)(\tan\theta_2+\lambda h_2)$ becomes $O(-\lambda)$ as $\lambda \to -\infty$), we can easily see that the eigenvalues mentioned in the theorem at least exist (figure 11). We have only to show that no other eigenvalue exists to complete the proof. This is obvious from the following lemma.

LEMMA 5. $\mathrm{d}f(\lambda)/\mathrm{d}\lambda \lesssim \mathrm{d}g(\lambda)/\mathrm{d}\lambda$ *for* $\lambda \gtrless 0$, *when* λ *is one of the eigenvalues of this problem.*

Proof. To prove this, first of all we will show that

$$\int_{y_1}^{y_2} \rho\varphi^2(y)\,\mathrm{d}y - h_1 p(y_1)\varphi^2(y_1) + h_2 p(y_2)\varphi^2(y_2) \gtrless 0 \qquad \text{(C 6)}$$

for $\lambda \gtrless 0$ if λ is one of the eigenvalues of this problem. Integrating (C 1) $\times \varphi(y)$ from $y = y_1$ to y_2 and using the boundary conditions at $y = y_1$, $y = y_2$, we obtain

$$\lambda\left[\int_{y_1}^{y_2} \rho\varphi^2(y)\,\mathrm{d}y - h_1 p(y_1)\varphi^2(y_1) + h_2 p(y_2)\varphi^2(y_2)\right]$$

$$= p(y_1)\tan\theta_1\varphi^2(y_1) - p(y_2)\tan\theta_2\varphi^2(y_2) + \int_{y_1}^{y_2}\left\{p(y)\left(\frac{\mathrm{d}\varphi(y)}{\mathrm{d}y}\right)^2 + q(y)\varphi^2(y)\right\}\mathrm{d}y > 0,$$

which gives (C 6). Then, using (C 5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{p(y_2)\,\partial\varphi(y,\lambda)/\partial y\,|_{y=y_2}}{\varphi(y_2,\lambda)}\right] \lessgtr p(y_2)\,h_2 = \frac{\mathrm{d}}{\mathrm{d}\lambda}[p(y_2)(\tan\theta_2 + \lambda h_2)].$$

Appendix D. Zero of U located at the boundary in limiting cases

We will discuss here the cases where a zero of U exists exactly at the boundary of the region in the limiting case; we must ascertain whether such a zero of U is in the interior or exterior region while the parameter remains finite.

The cases where U vanishes at the boundary consist of the following: the boundary condition becomes $V = 0$ in the limit $k \to 0$, $\omega \to$ finite; the boundary condition becomes $\mathrm{d}\eta/\mathrm{d}y = 0$ in the limit of $k \to 0$, $\omega \to 0$; the boundary condition becomes $\eta = 0$ in the limit $k \to \infty$, $|\omega| \to \infty$; and the boundary condition becomes $\mathrm{d}V/\mathrm{d}y = 0$ in the limit $k \to \infty$, $\omega \to$ finite. In fact, the following cases come under one of these conditions: $v = 0$ or $u = \pm\mathrm{i}v$ in the limit of $k \to 0$, $\omega \to$ finite and $\eta = 0$ or $H \to \infty$ in the limit of $k \to \infty$, $|\omega| \to \infty$.

We have only to compare the sign of $\mathrm{d}U/\mathrm{d}y$ at the boundary to the sign which U has there before the parameter reaches the limit, in order to judge the behaviour of such a zero approaching the boundary. If the signs of $U(y_1)$ and $\mathrm{d}U/\mathrm{d}y\,|_{y=y_1}$ are opposite, there is a zero of U approaching $y = y_1$ from the interior, and if they are the same there is no such zero in the interior, and vice versa for the boundary of $y = y_2$.

For the boundary condition $v = 0$ in the limit $k \to 0$, $\omega \to$ finite, for example, if we consider a small but finite k, the value of U is calculated from V and η as

$$U = (\mathrm{i}f/\omega)\,V + (kgH/\omega)\,\eta.$$

Since V vanishes exactly at the boundary, $U_{(boundary)}$ is equal to $(kgH/\omega)\,\eta_{(boundary)}$. Using also the relations of $U = (\mathrm{i}f/\omega)\,V$ and $\eta = (-1/\omega)\,\mathrm{d}/\mathrm{d}y(\mathrm{i}V)$, we obtain the result that a zero approaching $y = y_1$ exists if $\omega > 0$, and that a zero approaching $y = y_2$ exists if $\omega < 0$.

For the boundary condition $u = \pm\mathrm{i}v$ in the limit $k \to 0$, $\omega \to$ finite, since the relation $U = \pm\mathrm{i}V$ holds at the boundary, we can obtain, in the same way, the result that a zero approaching $y = y_1$ exists if $\omega > f(y_1)$, and that a zero approaching $y = y_2$ exists if $\omega < -f(y_2)$. Consequently, if the boundary condition at $y = y_1$ is $u = \mathrm{i}v$, the number of zeros of the modes leaps at $\omega = f(y_1)$; the mode with the eigenvalue superior and closest to $f(y_1)$ has two more zeros of U than that with the eigenvalue inferior and closest to $f(y_1)$. For this boundary condition, however, there is another special eigenfunction with the eigenvalue of $\omega = f(y_1)$, which fills this gap. If the boundary condition at $y = y_2$ is $u = -\mathrm{i}v$, likewise, the number of zeros of the modes leaps at $\omega = -f(y_2)$, but the other eigenfunction with the eigenvalue of $\omega = -f(y_2)$ fills this gap.

For the boundary condition $\eta = 0$ or $H \to \infty$ in the limit $k \to \infty$, $|\omega| \to \infty$, we obtain the result that a zero approaching $y = y_1$ exists if $\omega < 0$, and that a zero approaching $y = y_2$ exists if $\omega > 0$.

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